Lecture 4.

Finish off section on completeness by

<u>Thm 1.</u> Let (X, d) be a metric space.

(i) $X = \bigcup_{i \in I} C_i$, C_i are components. $I \leftarrow$ some index set

(ii) $C_i \cap C_j = \emptyset$, $i \neq j$.

(iii) Every C_i is closed.

(iv) If $X = G \subseteq \mathbb{C}$, open set, then C_i are open and I is countable.

More details in Lecture 3 notes.

Sequences & completeness

<u>Def.</u> ① If $\{x_n\}_{n=1}^{\infty}$ is sequence in X, then $\lim_{n \to \infty} x_n = x$ $(x_n \to x)$ if $d(x, x_n) \to 0$.

② $x \in X$ is <u>limit point</u> to $A \subseteq X$ if $\exists$ countable subset $\{x_1, \ldots, x_n, \ldots\} \subseteq A$

s.t. $x_n \to x$. Equivalently, $\exists$ seq. $\{x_n\}_{n=1}^{\infty}$ w/ distinct points s.t. $x_n \to x$.

Prop 1. Let $A \subseteq X$. Then,
A is closed $\iff$ A contains all its limit points.

Def ③ $\{x_n\}_{n=1}^{\infty}$ is a Cauchy seq. if $\forall \varepsilon > 0$ $\exists N$ s.t. $d(x_n, x_m) < \varepsilon$ when $n, m \geq N$.

④ X is complete if all Cauchy seq. converge.

Ex. ① $(\mathbb{R}^n, d_E)$ is complete. ($\Rightarrow \mathbb{C}$ is complete)

② $\mathbb{C}_\infty \cong S^2 \subseteq \mathbb{R}^3$ is complete. This follows from S^2 being closed in $\mathbb{R}^3$. In fact,

Prop 2. Let (X, d) be complete. Then,
$A \subseteq X$ is complete $\iff$ A is closed.

Pf. DIY.

Cantor's Thm. (X, d) is complete $\Leftrightarrow$ $\forall \{F_n\}_{n=1}^{\infty}$, $\emptyset \neq F_n \subseteq X$ closed s.t. $F_1 \supseteq F_2 \supseteq \dots$ and $\operatorname{diam} F_n \to 0$, it holds that $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.

$\operatorname{diam} F_n$: $\sup\{d(x,y) ; x, y \in F_n\}$.

Pf. $\Rightarrow$. Let $\{F_n\}_{n=1}^{\infty}$ be as above. Pick any $x_n \in F_n$, $n = 1, \dots, \infty$. Then, for $n, m \geq N$, $x_n, x_m \in F_N \Rightarrow d(x_n, x_m) \leq \operatorname{diam} F_N \to 0$. Thus, $\{x_n\}_{n=1}^{\infty}$ is Cauchy, so by completeness, $x_n \to x_0 \in X$. For any N, $x_n \in F_N$ for $n \geq N \Rightarrow x_0 \in F_N$ as F_N is closed. But N arbitrary $\Rightarrow$ $x_0 \in F_N$ $\forall N$ or $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Clearly, $\bigcap F_n$ cannot contain more than x_0 since $\operatorname{diam} F_n \to 0$.

$\Leftarrow$. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy seq. Let $F_n = \overline{\{x_n, x_{n+1}, \dots\}}$. $\{F_n\}_{n=1}^{\infty}$ satisfies assumptions in CT. $\Rightarrow \exists\, x_0 \in \bigcap_{n=1}^{\infty} F_n$. Now, $x_n \to x_0$ since $\operatorname{diam} F_n \to 0$. $\blacksquare$

Ex. 3 Why diam $F_n \to 0$? Consider

$$F_n = \{z \in \mathbb{C} : |z| \geq n\}.$$ This gives closed subsets $\mathbb{C} \supseteq F_1 \supseteq F_2 \supseteq \dots$ but $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

If we consider $F_n \subseteq \mathbb{C}_\infty$, then F_n are not closed. But if we add ∞, i.e. $\tilde{F}_n = F_n \cup \{\infty\} \subseteq \mathbb{C}_\infty$ and diam $\tilde{F}_n \to 0$. And $\infty \in \bigcap_{n=1}^{\infty} \tilde{F}_n$.

Rem. Both $\mathbb{C}$ and $\mathbb{C}_\infty$ are complete, but Ex 3 indicates a difference. $\mathbb{C}$ lacks "compactness".

Compactness.

Def. (5) A metric space (X,d) is compact if every open cover has a finite sub-cover, i.e. if $X \subseteq \bigcup_{i\in I} G_i$, G_i open, $\exists\, G_{i_1},\dots,G_{i_n}$ s.t. $X \subseteq \bigcup_{k=1}^{n} G_{i_k}$.

(As usual, $K \subseteq X$ is compact if (K,d) is compact.)

Prop 3. (i) $K \subseteq X$ compact $\Rightarrow$ K is closed.

(ii) $F \subseteq K$ closed, $K \subseteq X$ compact $\Rightarrow$ F compact.

(iii) X compact $\Rightarrow$ X complete.

(iv) X compact $\Rightarrow$ every seq. $\{x_n\}_{n=1}^{\infty}$ has a convergent subseq. $\{x_{n_k}\}_{k=1}^{\infty}$.

Property called seq. compactness.

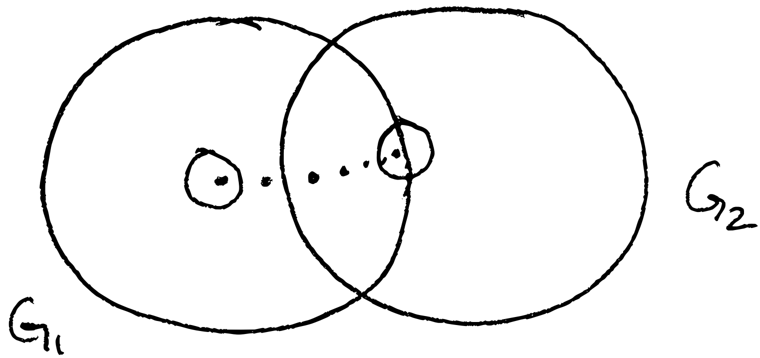
Pf. (i-ii). DIY.

(iii) We shall use Cantor. Let $\{F_n\}_{n=1}^{\infty}$ be a nested seq. of closed subsets of X as in CT. Suffices to show $\bigcap_n F_n \neq \emptyset$. Suppose $\bigcap_n F_n = \emptyset$. Then $G_n = X \setminus F_n$ is an open cover, so by compactness $\exists$ finite subcover $X = G_{n_1} \cup \ldots \cup G_{n_k}$, but then $F_{n_1} \cap \ldots \cap F_{n_k} = \emptyset$ which is a contradiction as $F_N \subseteq F_{n_1} \cap \ldots \cap F_{n_k}$ if $N \geq \max(n_1, \ldots, n_k)$. $\boxtimes$

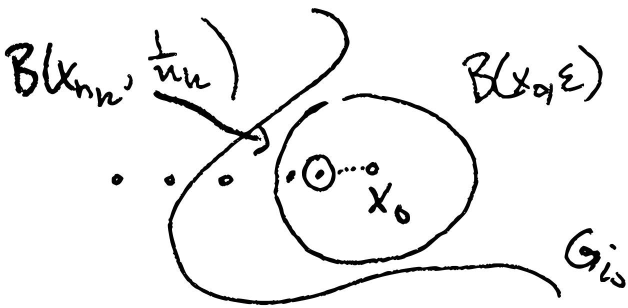
(iv) It is easy to see that seq. compactness $\Leftrightarrow$ Every infinite subset A has a limit point.
Let $A \subseteq X$ be infinite subset and X cpct. Must show A has a limit point. Suppose not. Then, $\forall x\ \exists\ B_x = B(x, \underset{\text{dep. on } x}{\varepsilon})$ s.t. $B_x \cap S$ contains at most <u>one point</u> (if $x \in S$ then $B_x \cap S = \{x\}$ if $x \notin S$ then $B_x \cap S = \emptyset$).

$\{B_x\}_{x\in \bar{X}}$ is an open cover of $\bar{X}$ $\Rightarrow$ $\exists x_1, \dots, x_n$ s.t. $\bar{X} = \bigcup_{k=1}^{n} B_{x_k}$, which contradicts $\# S = \infty$ since each B_{x_k} contains at most one pt of S. $\blacksquare$

<u>Lebesgue Covering Lemma</u>. Suppose (X,d) is seq. compact and $\{G_i\}_{i\in I}$ is an open cover. Then, $\exists \varepsilon > 0$ s.t. $\forall x \in X$, $B(x,\varepsilon) \subseteq G_i$ for some i.



Pf. Suppose not. Then, $\forall n \ \exists x_n$ s.t. $B(x_n, \frac{1}{n}) \not\subseteq G_i$ for any $i \in I$. By seq. compactness $\exists$ subseq. $x_{n_k} \to x_0$. But $x_0 \in G_{i_0}$, for some i_0, and by openness $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subseteq G_{i_0}$.



Clearly*, as $k \to \infty$, $x_{n_k} \to x_0$, $\frac{1}{n_k} \to 0$, for k suff. large $B(x_{n_k}, 1/n_k) \subseteq B(x_0, \varepsilon) \subseteq G_{i_0}$, which contradicts choice of x_{n_k}.

$\blacksquare$

Thm 2. TFAE for (X, d):

(i) $\overline{X}$ is compact.

(ii) $\overline{X}$ is seq. compact. (or every inf. set has a limit pt.)

(iii) $\overline{X}$ complete and totally bounded, i.e. $\forall \varepsilon > 0 \ \exists x_1, \dots, x_n \in \overline{X}$ s.t. $\overline{X} = \bigcup_{k=1}^{n} B(x_k, \varepsilon)$.

Pf. DIY

Heine-Borel Thm. $K \subseteq \mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded.

Pf. DIY.